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Convergence behavior of delayed cellular neural networks without periodic coefficients[☆]

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Abstract

In this work the convergence behaviors of delayed cellular neural networks without periodic coefficients are considered. Some new sufficient conditions are established to ensure that all solutions of the networks converge to a periodic function.
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1. Introduction

Let n correspond to the number of units in a neural network, $x_i(t)$ be the state vector of the ith unit at the time t, $a_{ij}(t)$ be the strength of the jth unit on the ith unit at time t, $b_{ij}(t)$ be the strength of the jth unit on the ith unit at time $t-\tau_{ij}(t)$, and $\tau_{ij}(t)\geq 0$ denote the transmission delay of the ith unit along the axon of the jth unit at the time t. It is well known that the delayed cellular neural networks are described by the following differential equations:

$$x_i'(t) = -c_i(t)h_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t), \quad i = 1, 2, \ldots, n, \tag{1.1}$$

for any activation functions of signal transmission f_j and g_j. Here $I_i(t)$ denotes the external bias on the ith unit at the time t, $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time t.

Since the cellular neural networks (CNNs) were introduced by Chua and Yang [1] in 1990, they have been successfully applied in signal and image processing, pattern recognition and optimization. Hence, CNNs have been the object of intensive analysis by numerous authors in recent years. In particular, extensive results on the problem of the existence and stability of periodic solutions for system (1.1) are given in many literature entries. We refer the reader to [2–8] and the references cited therein. Suppose that

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(H_0) $c_i, I_i, a_{ij}, b_{ij} : R \to R$ are continuous periodic functions, where $i, j = 1, 2, \ldots, n$.

Most authors of the bibliographies listed above obtained that all solutions of system (1.1) converge to a periodic function. However, to the best of our knowledge, few authors have considered the convergence behavior for all solutions of system (1.1) without the assumption (H_0). Thus, it is worthwhile to continue to investigate the convergence behavior for all solutions of system (1.1) in this case.

The main purpose of this work is to give new criteria for the convergence behavior for all solutions of system (1.1). By applying mathematical analysis techniques, without assuming (H_0), we derive some sufficient conditions ensuring that all solutions of system (1.1) converge to a periodic function, which are new and complement previously known results. Moreover, an example is also provided to illustrate the effectiveness of our results.

Consider the following delayed cellular neural networks:

$$x_i'(t) = -c_i^*(t)h_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}^*(t) f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij}^*(t) g_j(x_j\left(t - \tau_{ij}(t)\right)) + I_i^*(t), \tag{1.2}$$

where $i = 1, 2, \ldots, n$.

Throughout this work, for $i, j = 1, 2, \ldots, n$, it will be assumed that $c_i^*, I_i^*, a_{ij}^*, b_{ij}^*, \tau_{ij} : R \to R$ are continuous ω-periodic functions. Then, we can choose a constant τ such that

$$\tau = \max_{1\le i,j\le n} \left\{ \max_{t\in[0,\ \omega]} \tau_{ij}(t) \right\}. \tag{1.3}$$

We also assume that the following conditions hold.

(H_1) For each $i, j \in \{1, 2, \ldots, n\}$, $h_i \in C[R, R]$, and there exist nonnegative constants $\underline{d}_i$, $\tilde{L}_j$ and L_j such that

$$\underline{d}_i |u - v| \le (u - v)(h_i(u) - h_i(v)), \quad \text{for all } u, v \in R, \tag{1.4}$$

and

$$|f_j(u) - f_j(v)| \le \tilde{L}_j |u - v|, \qquad |g_j(u) - g_j(v)| \le L_j |u - v|, \quad \text{for all } u, v \in R. \tag{1.5}$$

(H_2) There exist constants $\eta > 0$, $\lambda > 0$ and $\xi_i > 0$, $i = 1, 2, \ldots, n$, such that for all $t > 0$, there holds

$$-[c_i^*(t)\underline{d}_i - \lambda]\xi_i + \sum_{j=1}^{n} |a_{ij}^*(t)| \tilde{L}_j \xi_j + \sum_{j=1}^{n} |b_{ij}^*(t)| \mathrm{e}^{\lambda\tau} L_j \xi_j < -\eta < 0, \quad i = 1, 2, \ldots, n.$$

(H_3) For any $i, j = 1, 2, \ldots, n$, $c_i, I_i, a_{ij}, b_{ij} : R \to R$ are continuous functions, and

$$\lim_{t\to+\infty} \left(c_i(t) - c_i^*(t)\right) = 0, \qquad \lim_{t\to+\infty} \left(I_i(t) - I_i^*(t)\right) = 0,$$
$$\lim_{t\to+\infty} \left(a_{ij}(t) - a_{ij}^*(t)\right) = 0, \qquad \lim_{t\to+\infty} \left(b_{ij}(t) - b_{ij}^*(t)\right) = 0.$$

The following lemma will be useful for proving our main results in Section 2.

Lemma 1.1 (*[7]*). *Let* (H_1) *and* (H_2) *hold. Then system* (1.2) *has exactly one ω-periodic solution.*

As usual, we introduce the phase space $C([-\tau, 0]; R^n)$ as a Banach space of continuous mappings from $[-\tau, 0]$ to R^n equipped with the supremum norm defined by

$$\|\varphi\| = \max_{1\le i\le n} \sup_{-\tau\le t\le 0} |\varphi_i(t)|$$

for all $\varphi = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T \in C([-\tau, 0]; R^n)$.

The initial conditions associated with system (1.1) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], i = 1, 2, \ldots, n, \tag{1.6}$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T \in C([-\tau, 0]; R^n)$.

For $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, we define the following norm:

$$\|Z(t)\|_\xi = \max_{i=1,2,\dots,n} \left|\xi_i^{-1} x_i(t)\right|.$$

2. Main results

Theorem 2.1. *Let* (H_1), (H_2) *and* (H_3) *hold. Suppose that* $Z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ *is the* ω*-periodic solution of system* (1.2). *Then, for every solution* $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ *of system* (1.1) *with any initial value* $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0]; R^n)$, *there holds*

$$\lim_{t\to+\infty} \left|x_i(t) - x_i^*(t)\right| = 0, \quad i = 1, 2, \dots, n.$$

Proof. Set

$$\delta_i(t) = -[c_i(t) - c_i^*(t)]h_i(x_i^*(t)) + \sum_{j=1}^{n} \left[a_{ij}(t) - a_{ij}^*(t)\right] f_j\left(x_j^*(t)\right) + \sum_{j=1}^{n} \left[b_{ij}(t) - b_{ij}^*(t)\right] g_j\left(x_j^*\left(t - \tau_{ij}(t)\right)\right) + \left[I_i(t) - I_i^*(t)\right],$$

where $i = 1, 2, \dots, n$. Since $Z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ is ω-periodic, together with (H_2) and (H_3), then $\forall\epsilon > 0$, we can choose a sufficient large constant $T > 0$ such that

$$|\delta_i(t)| < \frac{1}{4}\eta\epsilon, \quad \text{for all } t \ge T, \tag{2.1}$$

and

$$-[c_i(t)\underline{d}_i - \lambda]\xi_i + \sum_{j=1}^{n} \left|a_{ij}(t)\right| \tilde{L}_j \xi_j + \sum_{j=1}^{n} \left|b_{ij}(t)\right| \mathrm{e}^{\lambda\tau} L_j \xi_j < -\frac{1}{2}\eta < 0, \tag{2.2}$$

for all $t \ge T$, $i = 1, 2, \dots, n$.

Let $Z(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be a solution of system (1.1) with any initial value $\varphi = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in C([-\tau, 0];\ R^n)$, and define

$$u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T = Z(t) - Z^*(t).$$

Then

$$u_i'(t) = -c_i(t)(h_i(x_i(t)) - h_i(x_i^*(t))) + \sum_{j=1}^{n} a_{ij}(t)\left[f_j\left(x_j(t)\right) - f_j\left(x_j^*(t)\right)\right] + \sum_{j=1}^{n} b_{ij}(t)\left[g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - g_j\left(x_j^*\left(t - \tau_{ij}(t)\right)\right)\right] + \delta_i(t), \quad i = 1, 2, \dots, n. \tag{2.3}$$

Let i_t be such an index that

$$\xi_{i_t}^{-1}|u_{i_t}(t)| = \|u(t)\|_\xi\,. \tag{2.4}$$

Calculating the upper left derivative of $\mathrm{e}^{\lambda s}|u_{i_s}(s)|$ along (2.3), in view of (2.1) and (H_1), we have

$$D^+\left(\mathrm{e}^{\lambda s}|u_{i_s}(s)|\right)\Big|_{s=t} = \lambda\mathrm{e}^{\lambda t}|u_{i_t}(t)| + \mathrm{e}^{\lambda t}\mathrm{sign}(u_{i_t}(t))\left\{-c_{i_t}(t)(h_{i_t}(x_{i_t}(t)) - h_{i_t}(x_{i_t}^*(t))) + \sum_{j=1}^{n} a_{i_t j}(t)\right.$$

$$\times\left[f_j(x_j(t))-f_j(x_j^*(t))\right]+\sum_{j=1}^{n}b_{i_tj}(t)\left[g_j(x_j(t-\tau_{i_tj}(t)))-g_j(x_j^*(t-\tau_{i_tj}(t)))\right]+\delta_{i_t}(t)\Bigg\}$$

$$\le \mathrm{e}^{\lambda t}\left\{-\left[c_{i_t}(t)\underline{d}_{i_t}-\lambda\right]\left|u_{i_t}(t)\right|\xi_{i_t}^{-1}\xi_{i_t}+\sum_{j=1}^{n}a_{i_tj}(t)\tilde{L}_j\left|u_j(t)\right|\xi_j^{-1}\xi_j\right.$$

$$\left.+\sum_{j=1}^{n}b_{i_tj}(t)L_j|u_j(t-\tau_{i_tj}(t))|\xi_j^{-1}\xi_j\right\}+\frac{1}{4}\eta\epsilon\mathrm{e}^{\lambda t}. \tag{2.5}$$

Let

$$M(t)=\max_{-\tau\le s\le t}\left\{\mathrm{e}^{\lambda s}\|u(s)\|_\xi\right\}. \tag{2.6}$$

It is obvious that $\mathrm{e}^{\lambda t}\|u(t)\|_\xi\le M(t)$, and $M(t)$ is non-decreasing.

Now, we consider two cases.

Case (i). Suppose

$$M(t)>\mathrm{e}^{\lambda t}\|u(t)\|_\xi \quad \text{for all } t\ge T. \tag{2.7}$$

Then, we claim that

$$M(t)\equiv M(T) \text{ is a constant for all } t\ge T. \tag{2.8}$$

By way of contradiction, assume that (2.8) does not hold. Consequently, there exists $t_1>T$ such that $M(t_1)>M(T)$. We have

$$\mathrm{e}^{\lambda t}\|u(t)\|_\xi\le M(T) \quad \text{for all } -\tau\le t\le T.$$

So there must exist $\beta\in(T,t_1)$ such that

$$\mathrm{e}^{\lambda\beta}\|u(\beta)\|_\xi=M(t_1)\ge M(\beta),$$

which contradicts (2.7). This contradiction implies that (2.8) holds. It follows that there exists $t_2>T$ such that

$$\|u(t)\|_\xi\le\mathrm{e}^{-\lambda t}M(t)=\mathrm{e}^{-\lambda t}M(T)<\epsilon \quad \text{for all } t\ge t_2. \tag{2.9}$$

Case (ii). If there is such a point $t_0\ge T$ that $M(t_0)=\mathrm{e}^{\lambda t_0}\|u(t_0)\|_\xi$, then, using Eqs. (2.1), (2.2) and (2.5), we get

$$D^+\left(\mathrm{e}^{\lambda s}|u_{i_s}(s)|\right)\Big|_{s=t_0}$$

$$\le\left\{-[c_{i_{t_0}}(t_0)\underline{d}_{i_{t_0}}-\lambda]\mathrm{e}^{\lambda t_0}|u_{i_{t_0}}(t_0)|\xi_{i_{t_0}}^{-1}\xi_{i_{t_0}}+\sum_{j=1}^{n}a_{i_{t_0}j}(t_0)\tilde{L}_j\mathrm{e}^{\lambda t_0}|u_j(t_0)|\xi_j^{-1}\xi_j\right.$$

$$\left.+\sum_{j=1}^{n}b_{i_{t_0}j}(t_0)L_j\mathrm{e}^{\lambda(t_0-\tau_{i_{t_0}j}(t_0))}|u_j(t_0-\tau_{i_{t_0}j}(t_0))|\xi_j^{-1}\mathrm{e}^{\lambda\tau_{i_{t_0}j}(t_0)}\xi_j\right\}+\frac{1}{4}\eta\epsilon\mathrm{e}^{\lambda t_0}$$

$$\le\left\{-[c_{i_{t_0}}(t_0)\underline{d}_{i_{t_0}}-\lambda]\xi_{i_{t_0}}+\sum_{j=1}^{n}a_{i_{t_0}j}(t_0)\tilde{L}_j\xi_j+\sum_{j=1}^{n}b_{i_{t_0}j}(t_0)\mathrm{e}^{\lambda\tau}L_j\xi_j\right\}M(t_0)+\frac{1}{4}\eta\epsilon\mathrm{e}^{\lambda t_0}$$

$$<-\frac{1}{2}\eta M(t_0)+\frac{1}{2}\eta\epsilon\mathrm{e}^{\lambda t_0}. \tag{2.10}$$

In addition, if $M(t_0)\ge\epsilon\mathrm{e}^{\lambda t_0}$, then $M(t)$ is strictly decreasing in a small neighborhood $(t_0,t_0+\delta_0)$. This contradicts that $M(t)$ is non-decreasing. Hence,

$$\mathrm{e}^{\lambda t_0}\|u(t_0)\|_\xi=M(t_0)<\epsilon\mathrm{e}^{\lambda t_0}, \quad \text{and} \quad \|u(t_0)\|_\xi<\epsilon. \tag{2.11}$$

Furthermore, for any $t>t_0$, by the same approach as was used in the proof of (2.11), we have

$$\mathrm{e}^{\lambda t}\|u(t)\|_\xi<\epsilon\mathrm{e}^{\lambda t},\ \text{and}\ \|u(t)\|_\xi<\epsilon, \quad \text{if } M(t)=\mathrm{e}^{\lambda t}\|u(t)\|_\xi. \tag{2.12}$$

On the other hand, if $M(t) > \mathrm{e}^{\lambda t}\|u(t)\|_{\xi}$, $t > t_0$. We can choose $t_0 \le t_3 < t$ such that

$$M(t_3) = \mathrm{e}^{\lambda t_3}\|u(t_3)\|_{\xi}, \quad \|u(t_3)\|_{\xi} < \epsilon \quad \text{and} \quad M(s) > \mathrm{e}^{\lambda s}\|u(s)\|_{\xi} \quad \text{for all } s \in (t_3, t].$$

Using an argument similar to that in the proof of **Case (i)**, we can show that

$$M(s) \equiv M(t_3) \text{ is a constant for all } s \in (t_3, t], \tag{2.13}$$

which implies that

$$\|u(t)\|_{\xi} < \mathrm{e}^{-\lambda t}M(t) = \mathrm{e}^{-\lambda t}M(t_3) = \|u(t_3)\|_{\xi}\mathrm{e}^{-\lambda(t-t_3)} < \epsilon.$$

In summary, there must exist $N > 0$ such that $\|u(t)\|_{\xi} \le \epsilon$ holds for all $t > N$. This implies that the proof of Theorem 2.1 is now complete. □

3. An example

In this section, we give an example to demonstrate the results obtained in previous sections.

Example 3.1. Consider the following CNNs with time-varying delays:

$$\begin{cases} x_1'(t) = -\left(1 - \dfrac{2}{1+|t|}\right)x_1(t) + \left(\dfrac{1}{4} + \dfrac{t}{1+t^2}\right)f_1(x_1(t)) + \left(\dfrac{1}{36} + \dfrac{2t}{1+t^2}\right)f_2(x_2(t)) \\ \quad + \left(\dfrac{1}{4} + \dfrac{t}{2+t^2}\right)g_1\left(x_1\left(t - \sin^2 t\right)\right) + \left(\dfrac{1}{36} + \dfrac{4t}{1+t^2}\right)g_2\left(x_2\left(t - 2\sin^2 t\right)\right) \\ \quad + \left(\cos t + \dfrac{t}{1+t^2}\right), \\ x_2'(t) = -\left(1 - \dfrac{4}{1+2|t|}\right)x_2(t) + \left(1 + \dfrac{t}{1+t^2}\right)f_1(x_1(t)) + \left(\dfrac{1}{4} + \dfrac{5t}{1+t^2}\right)f(x_2(t)) \\ \quad + \left(1 + \dfrac{t}{1+6t^2}\right)g_1\left(x_1\left(t - 5\sin^2 t\right)\right) + \left(\dfrac{1}{4} + \dfrac{t}{8+t^2}\right)g_2\left(x_2\left(t - \sin^4 t\right)\right) + \left(\sin t + \dfrac{t}{1+t^6}\right), \end{cases} \tag{3.1}$$

where $f_1(x) = f_2(x) = g_1(x) = g_2(x) = \arctan x$.

Note the following CNNs:

$$\begin{cases} x_1'(t) = -x_1(t) + \dfrac{1}{4}f_1(x_1(t)) + \dfrac{1}{36}f_2(x_2(t)) + \dfrac{1}{4}g_1(x_1(t - \sin^2 t)) + \dfrac{1}{36}g_2(x_2(t - 2\sin^2 t)) + \cos t, \\ x_2'(t) = -x_2(t) + f_1(x_1(t)) + \dfrac{1}{4}f(x_2(t)) + g_1(x_1(t - 5\sin^2 t)) + \dfrac{1}{4}g_2(x_2(t - \sin^4 t)) + \sin t, \end{cases} \tag{3.2}$$

where

$$c_1^*(t) = c_2^*(t) = L_1 = L_2 = \tilde{L}_1 = \tilde{L}_2 = 1, \qquad a_{11}^*(t) = b_{11}^*(t) = \frac{1}{4}, \qquad a_{12}^*(t) = b_{12}^*(t) = \frac{1}{36},$$
$$a_{21}^*(t) = b_{21}^*(t) = 1, \qquad a_{22}^*(t) = b_{22}^*(t) = \frac{1}{4}, \qquad \tau = 5.$$

Then, we get

$$d_{ij} = \frac{1}{c^*(t)_i}(a_{ij}^*(t)\tilde{L}_j + b_{ij}^*(t)L_j) \quad i, j = 1, 2, \quad D = (d_{ij})_{2\times 2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{18} \\ 2 & \frac{1}{2} \end{pmatrix}.$$

Hence, we have $\rho(D) = \frac{5}{6} < 1$. Therefore, it follows from the theory of the M-matrix in [9] that there exist constants $\bar{\eta} > 0$ and $\xi_i > 0$, $i = 1, 2$, such that for all $t > 0$, there holds

$$-c_i^*(t)\xi_i + \sum_{j=1}^{n}|a_{ij}^*(t)|\tilde{L}_j\xi_j + \sum_{j=1}^{n}|b_{ij}^*(t)|L_j\xi_j < -\bar{\eta} < 0, \quad i = 1, 2.$$

Then, we can choose constants $\eta > 0$ and $0 < \lambda < 1$ such that

$$-[c_i^*(t) - \lambda]\xi_i + \sum_{j=1}^{n} |a_{ij}^*(t)|\tilde{L}_j\xi_j + \sum_{j=1}^{n} |b_{ij}^*(t)|e^{\lambda\tau} L_j\xi_j < -\eta < 0, \quad i = 1, 2, \forall t > 0,$$

which implies that system (3.1) and (3.2) satisfies (H_1), (H_2) and (H_3). Hence, from Lemma 1.1 and Theorem 2.1, system (3.2) has exactly one 2π-periodic solution. Moreover, all solutions of system (3.1) converge to the periodic solution of system (3.2).

Remark 3.1. Since CNN (3.1) is a very simple form of a delayed neural networks without periodic coefficients, therefore all the results in [2–8] and the references therein are not applicable for proving that all solutions of system (3.1) converge to a periodic function. This implies that the results of this work are essentially new.

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